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SOME NOTES ABOUT A CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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This paper is dedicated to the 70th anniversary of Professor Srivastava

Abstract

The object of this paper is to obtain sharp results involving coefficient bounds, growth and distortion properties for some classes of analytic and univalent functions with negative coefficients.

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1. Introduction and definitions

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and *univalent* in the unit disk E. We denote by C and S^* the classes of convex and starlike functions, respectively.

A function $f(z)$ analytic in E, is said to be *starlike* of order β $(0 \leq \beta < 1)$ in E if $f(0) = f'(0) - 1 = 0$ and for $z \in E$

$$\Re \frac{zf'(z)}{f(z)} > \beta$$

The class of such functions will be denoted by S^*_β. Clearly, $S^*_0 = S^*$.

A function $f(z)$ analytic in E is said to be *close-to-convex* of order β $(0 \leq \beta < 1)$ in E if there exists a function $g(z) \in S^*$ and a real number γ such that, for $z \in E$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Re e^{i\gamma} \frac{zf'(z)}{g(z)} > \beta.$$

The class of such functions is denoted by K_β.

A function $f(z)$ is said to be *close-to-star* of order β $(0 \leq \beta < 1)$ if there exists a function $g(z) \in S^*$ such that, for $z \in E$

$$\Re \frac{f(z)}{g(z)} > \beta.$$

The class of such functions will be denoted by R_β.

A function $f(z)$, analytic in E with $f(0) = f'(0) - 1 = 0$ is said to be *quasi-convex* if and only if there exists a function $g(z) \in C$ such that for $z \in E$

$$\Re \frac{(zf'(z))'}{g'(z)} > \beta.$$

The class of such functions will be denoted by C^*_β.

Let T denotes the subclass of S, consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

We denote $T^*_\beta = S^*_\beta \cap T$; $K^*_\beta = K_\beta \cap T$; $R^*_\beta = R_\beta \cap T$; $L^*_\beta = C^*_\beta \cap T$.

It is known that $T = T^*_0 = T^*$ and $f \in T^*_\beta$ if, and only if, for $0 \leq \beta < 1$

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| \leq 1 \qquad [1].$$

In [2] Schild considered a subclass of T consisting of polynomials having $|z| = 1$ as a radius of univalence. Schild showed ([2]) that a necessary and sufficient condition for $f \in T$ is

$$1 - \sum_{n=2}^{\infty} n|a_n| = 0.$$

With the aid of this result he derived better results for certain quantities connected with conformal mapping of univalent functions. Other subclasses of T have been studied by Gupta and Jain [1], [2] and Silverman [5], [6].

In this paper we consider the following subclass $H_{t,\alpha}(\beta)$ of T:

Definition . A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is said to be in $H_{t,\alpha}(\beta)$ $(0 \le \alpha < 1,\ 0 \le \beta < 1,\ 0 < t \le 1)$, if there exists a function $g \in T^*$, with

$$g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$$

such that for $z \in E$

$$\Re \left\{ \frac{tzf'(z) + (1-t)z\,(zf'(z))'}{\alpha g(z) + (1-\alpha) z g'(z)} \right\} > \beta. \tag{1}$$

Evidently, $H_{1,1}(\beta) = K^*_\beta$, the class of close-to-convex functions of order β introduced by [7]. Note also that $H_{1,0}(\beta) = R^*_\beta$ and $H_{0,1}(\beta) = L^*_\beta$.

In the sequel we write

$$J_{t,\alpha}(f,g,z_0) = \frac{1}{1-\beta} \left\{ \frac{tz_0 f'(z_0) + (1-t) z_o (z_o f'(z_0))'}{\alpha g(z_0) + (1-\alpha) z_0 g'(z_0)} - \beta \right\}. \tag{2}$$

2. Some results about the class $H_{t,\alpha}(\beta)$

Lemma . *Let $f \in H_{t,\alpha}(\beta)$ be given by (1.1). Then*

$$\min_{|z| \le r < 1} \Re J_{t,\alpha}(f,g,z) = J_{t,\alpha}(f,g,r).$$

The proof of this lemma is standard.

Theorem 1. *Let $f(z) \in H_{t,\alpha}(\beta)$ be given by (1.1). Then, for $0 < r < 1$,*

$$\begin{aligned} &t \sum_{n=2}^{\infty} \frac{[n|a_n| - (\alpha + n(1-\alpha))|b_n|]\, r^{n-1}}{1 - \sum_{n=2}^{\infty} (\alpha + n(1-\alpha))|b_n| r^{n-1}} + \\ &+(1-t) \frac{\sum_{n=2}^{\infty} \left[n^2 |a_n| - (\alpha + n(1-\alpha))|b_n| \right] r^{n-1}}{1 - \sum_{n=2}^{\infty} (\alpha + n(1-\alpha))|b_n| r^{n-1}} < 1 - \beta \end{aligned} \tag{3}$$

when $0 \le \alpha \le 1$, $0 \le t \le 1$. The estimate (3) is also sufficient for f to be in $H_{t,\alpha}(\beta)$.

Remark A. If $\sum_{n=2}^{\infty} n|a_n| < 1$, $\sum_{n=2}^{\infty} n|b_n| < 1$ and $\sum_{n=2}^{\infty} n^2|a_n| < \infty$, then $J_{t,\alpha}(f,g,r)$ is continuous at $r = 1$ and (2.1) may be replaced by

$$\sum_{n=2}^{\infty}\left(tn + (1-t)n^2\right)|a_n| - \beta\sum_{n=2}^{\infty}(\alpha + (1-\alpha)n)|b_n| \le 1-\beta. \tag{4}$$

Remark B. In [5] it was shown that $\sum_{n=2}^{\infty}|b_n| \le \frac{1}{2}$ for $g \in T^*$, so that

$$\sum_{n=2}^{\infty}\left(tn + (1-t)n^2\right)|a_n| \le 1-\beta+\frac{\beta(2-\alpha)}{2} = 1-\frac{\alpha\beta}{2}. \tag{5}$$

In fact, (5) is a necessary condition for f to be in $H_{t,\alpha}(\beta)$ and we could always take $g(z) = z - \frac{1}{2}z^2$ and (5) would also be sufficient.

Theorem 2. *Let $f \in H_{t,\alpha}(\beta)$ be given by (1.1). Then*

$$a_n \le A_n = \frac{\alpha\beta + (1-\alpha\beta)n}{n^2\left[t + (1-t)n\right]}.$$

The result is sharp for every n, with equality for

$$f(z) = z - A_n z^n$$

and $g \in T$, with $g(z) = z - \frac{1}{n}z^n$.

Theorem 3. *If $f \in H_{t,\alpha}(\beta)$*

$$r - \frac{(2-\alpha\beta)}{4(2-t)}r^2 \le |f(z)| \le r + \frac{(2-\alpha\beta)}{4(2-t)}r^2, \qquad |z| \le r$$

$$1 - \frac{(2-\alpha\beta)}{2(2-t)}r \le |f'(z)| \le 1 + \frac{(2-\alpha\beta)}{2(2-t)}r, \qquad |z| \le r.$$

Equality holds in all cases for

$$f(z) = z - \frac{2-\alpha\beta}{4(2-t)}z^2.$$

3. Additional results for the class $H_{t,\alpha}(\beta)$

Theorem 4. *The family $H_{t,\alpha}(\beta)$ is convex.*

P r o o f. We modify the method of Silvia and Silverman [7].

Suppose that f_1 and $f_2 \in H_{t,\alpha}(\beta)$, with $f_1(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $f_2(z) = z - \sum_{n=2}^{\infty} |c_n| z^n$ with respect to g_1 and $g_2 \in T$, with $g_1 = z - \sum_{n=2}^{\infty} |b_n| z^n$ and $g_2 = z - \sum_{n=2}^{\infty} |d_n| z^n$.

We will show, for $0 \leq \lambda \leq 1$, that

$$\lambda f_1(z) + (1-\lambda) f_2(z) = z - \sum_{n=2}^{\infty} \gamma_n(\lambda) z^n \in H_{t,\alpha}(\beta)$$

with respect to

$$\lambda g_1(z) + (1-\lambda) g_2(z) = z - \sum_{n=2}^{\infty} \delta_n(\lambda) z^n \in T$$

where $\gamma_n(\lambda) = \lambda |a_n| + (1-\lambda)|c_n|$ and $\delta_n(\lambda) = \lambda |b_n| + (1-\lambda)|d_n|$.

Since

$$\begin{aligned}
& \sum_{n=2}^{\infty} \left\{ \left[tn + (1-t)n^2 \right] \gamma_n(\lambda) - \beta \left[(1-\alpha)n + \alpha \right] \delta_n(\lambda) \right\} \\
= & \sum_{n=2}^{\infty} \left\{ \left[tn + (1-t)n^2 \right] \left[\lambda |a_n| + (1-\lambda)|c_n| \right] \right. \\
& \left. \beta \left[(1-\alpha)n + \alpha \right] \left[\lambda |b_n| + (1-\lambda)|c_n| \right] \right\} \\
= & \lambda \sum_{n=2}^{\infty} \left\{ \left[tn + (1-t)n^2 \right] |a_n| - \beta \left[(1-\alpha)n + \alpha \right] |b_n| \right\} \\
& (1-\lambda) \sum_{n=2}^{\infty} \left\{ \left[tn(1-t)n^2 \right] |c_n| - \beta \left[(1-\alpha)n + \alpha \right] |d_n| \right\} \\
\leq & 1 - \beta,
\end{aligned}$$

then the result follows. ∎

THEOREM 5. *If $f(z) \in H_{t,\alpha}(\beta)$, then $f(z)$ is convex in the disc $|z| < r = r(t,\alpha,\beta)$ where*

$$r(t,\alpha,\beta) = \inf_n \left\{ \frac{t + (1-t)n}{\alpha\beta + (1-\alpha\beta)n} \right\}^{\frac{1}{n-1}}, \qquad n = 2, 3, \ldots$$

This result is sharp. The extremal function is

$$f(z) = z - \frac{\alpha\beta + (1-\alpha\beta)n}{n^2[t + (1-t)n]} z^n.$$

P r o o f. It suffices to show that $\left|\frac{zf''(z)}{f'(z)}\right| \le 1$ for $|z| < 1$.

First we note that

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{n=2}^{\infty} n(n-1)|a_n|z^{n-1}}{1-\sum\limits_{n=2}^{\infty} n|a_n|z^{n-1}}\right| \le \frac{\sum\limits_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} n|a_n||z^{|}n-1}.$$

Thus $\left|\frac{zf''(z)}{f'(z)}\right| \le 1$ whenever

$$\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1} \le 1.$$

But in view of Remark A [3] we have

$$\sum_{n=2}^{\infty} \left[tn + (1-t)n^2\right]|a_n| \le \frac{\alpha\beta + (1-\alpha\beta)n}{n} \tag{6}$$

where we have used the fact that $|b_n| \le \frac{1}{n}$ [5].

Hence, $f(z)$ is convex if

$$n^2|z|^{n-1} \le \frac{n^2\left[t + (1-t)n^2\right]}{\alpha\beta + (1-\alpha\beta)n}$$

that is

$$|z| \le \left\{\frac{t + (1-t)n}{\alpha\beta + (1-\alpha\beta)n}\right\}^{\frac{1}{n-1}}.$$

Thus the radius of convexity is given by

$$r(t,\alpha,\beta) = \inf_n \left\{\frac{t + (1-t)n}{\alpha\beta + (1-\alpha\beta)n}\right\}^{\frac{1}{n-1}}.$$

■

THEOREM 6. *If* $f_1(z) = z - \sum\limits_{n=2}^{\infty} |a_n|z^n$ *and* $f_2(z) = z - \sum\limits_{n=2}^{\infty} |c_n|z^n$ *are in* $H_{t,\alpha}(\beta)$, *then*

$$h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty} |a_n + c_n|z^n$$

is also in $H_{t,\alpha}(\beta)$.

P r o o f. The proof follows directly by appealing to Theorem 1, [3]. In fact, if $f_1(z)$ and $f_2(z)$ belong to $H_{t,\alpha}(\beta)$, then from (6) we have

$$\sum_{n=2}^{\infty}\left[tn+(1-t)n^2\right]|a_n| \leq \frac{\alpha\beta+(1-\alpha\beta)n}{n} \tag{7}$$

and

$$\sum_{n=2}^{\infty}\left[tn+(1-tn)^2\right]|c_n| \leq \frac{\alpha\beta+(1-\alpha)n}{n}. \tag{8}$$

For $h(z)$ to be in $H_{t,\alpha}(\beta)$, it is sufficient to show that

$$\frac{1}{2}\sum_{n=2}^{\infty}\left[tn+1(1-t)n^2\right]|a_n+c_n| \leq \frac{\alpha\beta+(1-\alpha\beta)n}{n}$$

which follows immediately on using (7) and (8). ∎

THEOREM 7. *Let* $f_1(z) = z$ *and* $f_n(z) = \dfrac{\alpha\beta+(1-\alpha\beta)n}{n^2\left[t+(1-t)n\right]}z^n$. *Then* $f(z) \in H_{t,\alpha}(\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$$

where $\lambda_n \geq 0$ *and* $\sum\limits_{n=1}^{\infty}\lambda_n = 1$.

P r o o f. Suppose that $f(z) = \sum\limits_{n=1}^{\infty}\lambda_n f_n(z) = z\lambda_1 + \left[z - \dfrac{z-\alpha\beta}{4(2-t)}t^2\right]\lambda_2 + \ldots + \left[z - \dfrac{\alpha\beta+(1-\alpha\beta)n}{n^2\left[t+(1-t)n\right]}z^n\right]\lambda_n = z - \sum\limits_{n=2}^{\infty}\dfrac{\alpha\beta+(1-\alpha\beta)n}{n^2\left[t+(1-t)n\right]}\lambda_n z^n$. Then

$$\sum_{n=2}^{\infty}\left\{n\left[t+(1-t)n\right]\lambda_n\frac{\alpha\beta+(1-\alpha\beta)n}{n^2\left[t+(1-t)n\right]}\right\} \leq \frac{\alpha\beta+(1-\alpha\beta)n}{n}.$$

Thus, by Theorem 1 and Remark A, $f(z) \in H_{t,\alpha}(\beta)$.

Conversely, suppose $f(z) \in H_{t,\alpha}(\beta)$. By Theorem 2, we have

$$|a_n| \leq \frac{\alpha\beta+(1-\alpha\beta)n}{n^2\left[t+(1-t)n\right]}.$$

Setting

$$\lambda_n = \frac{n^2\left[t+(1-t)n\right]}{\alpha\beta+(1-\alpha\beta)n}|a_n|$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n,$$

we have that

$$f(z) = \left(\lambda_1 + \sum_{n=2}^{\infty} \lambda_n\right) z - \sum_{n=2}^{\infty} \frac{\alpha\beta + (1-\alpha\beta)n}{n^2\left[t + (1-t)n\right]} \lambda_n z^n$$

$$= z\lambda_1 + \left[z - \frac{2-\alpha\beta}{4(2-t)} z^2\right] \lambda_2 + \ldots$$

$$+ \left[z - \frac{\alpha\beta + (1-\alpha\beta)n}{[t + (1-t)n]} z^n\right] \lambda_n = \sum_{n=1}^{\infty} \lambda_n f_n(z).$$

This completes the proof of the theorem. The extreme points of $H_{t,\alpha}(\beta)$ are functions f_1 and f_2. ■

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